STAT 3743: Probability and Statistics

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Probability

- Random experiment: outcome not known in advance
- Sample space: set of all possible outcomes (S)
- Probability related to Set Theory
  - subsets A, B, C, *etc.* are *events*
  - $\emptyset$ represents the empty set

How to do it with R

```
> library(prob)
> S <- data.frame(lands = c("down",
+      "up", "side"))
> S <- tosscoin(3)
```

Set Theory review

Name	Denoted	Defined by elements	R syntax
Union	$A \cup B$	in A or B or both	union(A, B)
Intersection	$A \cap B$	in both A and B	intersect (A, B)
Difference	$A \backslash B$	in A but not in B	setdiff (A, B)
Complement	A^c	in S but not in A	setdiff (S, A)

Table: Set operations

Algebra of sets

- $A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset, A \cup S = S, \quad A \cap S = A, \ldots$
- Commutative property:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

- Associative property:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive property:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup B), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap B)$$

Example

Write "neither A nor B occurs"

Example

"A occurs, but not B"

Example

"A or B occurs, but not both"

Definition

The sets A and B are *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$. We say $A_1, A_2, \ldots, A_k$ are m.e. if $A_i \cap A_j = \emptyset$ when $i \neq j$.

- Have all kinds of events, want to know chance of an event A
- The probability of A is the proportion of times that A occurs in repeated trials of a random experiment as the number of trials increases without bound.

Axioms for Probability

Axiom 1.

$\mathbb{P}(A) \geq 0$ for any event $A \subset S$.

Axiom 2.

$\mathbb{P}(S) = 1$.

Axiom 3.

If the events $A_1, A_2, A_3 \ldots$ are disjoint then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Properties of probability

Property 1.

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

Property 2.

$\mathbb{P}(\emptyset) = 0$

Property 3.

If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Properties of probability

Property 4.

$0 \leq \mathbb{P}(A) \leq 1$

Property 5. (General Addition Rule)

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Properties of probability

What about 3 events?

Corollary. (Boole's Inequality)

$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$

How do we assign probabilities?

Finite sample space

$$S = \{e_1,\ e_2,\ \ldots,\ e_N\}$$

Need

1. $p_i \geq 0$
2. $\mathbb{P}(S) = \sum_{i=1}^{N} p_i = 1$

Equally likely outcomes
means

$$p_1 = p_2 = \cdots = p_N = p \quad \Longrightarrow \quad p = 1/N$$

How do we assign probabilities?

Given $A \subset S$, write

$$A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$$

Then

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(a_{i_1}) + \mathbb{P}(a_{i_2}) + \cdots + \mathbb{P}(a_{i_k}), \\
&= \frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N}, \\
&= \frac{k}{N} = \frac{\#(A)}{\#(S)}.
\end{aligned}$$

Examples

Example 1.

Toss a coin

Example 2.

Toss 2 coins

$\mathbb{P}(\text{at least 1 head}) =$

$\mathbb{P}(\text{no heads}) =$

Examples

Example 3.

Three child family

$\mathbb{P}(\text{exactly 2 boys}) =$

$\mathbb{P}(\text{at most 2 boys}) =$

Example 4.

Roll a die

Examples

Example 5.

Deck of cards. Select 1 card at random.

$A \longrightarrow \{\text{Ace}\} \qquad \mathbb{P}(A) =$

$B \longrightarrow \{\text{Clubs}\} \qquad \mathbb{P}(B) =$

$\mathbb{P}(A \cap B) =$

$\mathbb{P}(A \cup B) =$

Examples

Example 6.

Poker hand $\longrightarrow$ STUD poker

$$S = \{ \qquad\qquad\qquad\qquad \}$$

$\mathbb{P}(\text{Royal Flush}) =$

How to count

Multiplication Principle.

An experiment has two steps. First step can be done in n_1 ways, Second step can be done in n_2 ways. The whole experiment may be done in

$$n_1 n_2 \text{ ways}$$

If it has k steps which can be done in $n_1, n_2, \ldots, n_k$ ways, then the whole experiment may be done in

$$n_1 n_2 \cdots n_k \text{ ways}$$

Examples

Examples.

1. Want to eat a pizza

2. Toss 6 coins

3. Roll 112 dice

   What about $\mathbb{P}(70 \text{ sixes})$?

How to count

Theorem.
The number of ways to select an *ordered* sample of k subjects from a population that has n distinguishable members is

- n^k if sampling is done with replacement,
- $n(n-1)(n-2)\cdots(n-k+1)$ if sampling is done without replacement.

Here, ORDER is IMPORTANT

Examples

Examples.

1. Flip a coin 7 times
2. 20 students, select president, vice-president, treasurer
3. Rent 5 movies. Want to watch 3 movies on the first night.

How to count

Theorem.
The number of ways to select an *unordered* sample of k subjects from a population that has n distinguishable members is

- $(n-1+k)!/[(n-1)!k!]$ if sampling is done with replacement,
- $n!/[k!(n-k)!]$ if sampling is done without replacement.

$n!/[k!(n-k)!]$ is a *binomial coefficient* "n choose k"

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

More about binomial coefficients

Birthday problem

- n people in a class
- 365 days/year, equally likely
- $\mathbb{P}$(at least two have same birthday)

$$1 - \frac{\#(A)}{\#(S)} =$$

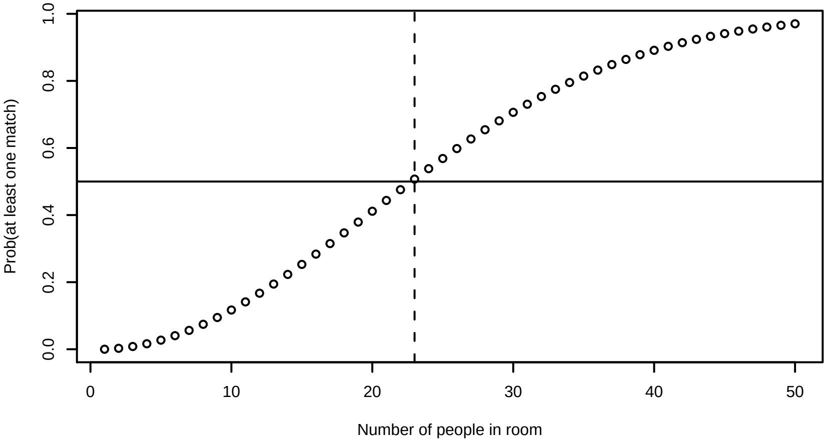

Figure: The birthday problem

Poker hands

52 cards $\longrightarrow$ 5 card hand

$$S = \{\text{all possible 5 card hands}\}$$

(should shuffle times???)

$$A = \text{Royal Flush} = \{A,\ K,\ Q,\ J,\ \text{all same suit}\}$$

$$B = \{\text{Four of a kind}\}$$

Conditional probability

52 cards $\longrightarrow$ draw 2 cards (without replacement)

$$A = \{\text{1st card drawn is Ace}\}$$

$$B = \{\text{2nd card drawn is Ace}\}$$

Then

$$\mathbb{P}(A) =$$

$$\mathbb{P}(B) = \Bigg\{$$

Conditional probability

Definition.

The conditional probability of B given that the event A occurred is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}, \quad \text{if } \mathbb{P}(A) > 0.$$

Conditional probability

Example

Toss a coin twice.

$$A = \{\text{a head occurs}\}$$

$$B = \{\text{a head and tail occurs}\}$$

- $\mathbb{P}(A|B) =$
- $\mathbb{P}(B|A) =$

Conditional probability

Example

Toss a die twice.

$$A = \{\text{outcomes match}\}$$

$$B = \{\text{sum of outcomes} \geq 8\}$$

- $\mathbb{P}(A) =$
- $\mathbb{P}(B) =$
- $\mathbb{P}(A \cap B) =$
- $\mathbb{P}(A|B) =$
- $\mathbb{P}(B|A) =$

Properties

Note.

For any fixed event A with $\mathbb{P}(A) > 0$,

1. $\mathbb{P}(B|A) \geq 0$, for all events $B \subset S$,
2. $\mathbb{P}(S|A) = 1$, and
3. If B_1, B_2, B_3,... are disjoint events, then

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} B_k \,\middle|\, A\right) = \sum_{k=1}^{\infty} \mathbb{P}(B_k|A).$$

More properties

Note.

For any events A, B, and C with $\mathbb{P}(A) > 0$,

1. $\mathbb{P}(B^c|A) = 1 - \mathbb{P}(B|A)$.
2. If $B \subset C$ then $\mathbb{P}(B|A) \leq \mathbb{P}(C|A)$.
3. $\mathbb{P}[(B \cup C)|A] = \mathbb{P}(B|A) + \mathbb{P}(C|A) - \mathbb{P}[(B \cap C|A)]$.
4. For any two events A and B,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B|A).$$

For 3 events:

Conditional probability

Example.

Recall the aces problem

$$A = \{\text{1st card drawn is Ace}\}$$

$$B = \{\text{2nd card drawn is Ace}\}$$

- $\mathbb{P}(\text{both Aces}) =$

Conditional probability

Example.

Urn with 10 balls, 7 red and 3 green. Select 3 balls successively from the urn.

$$A = \{\text{1st ball red}\}$$

$$B = \{\text{2nd ball red}\}$$

$$C = \{\text{3rd ball red}\}$$

- $\mathbb{P}(\text{all red}) =$

Good example

Two urns. First: 5 red, 3 green. Second: 2 red, 6 green

1 ball transferred. Select 1 ball.

$\mathbb{P}(\text{red}) =$

What if you don't look?

$\mathbb{P}(\text{second card is Ace}) =$

Good example (continued)

$\mathbb{P}(\text{red}) =$

What if you don't look?

$\mathbb{P}(\text{second card is Ace}) =$

Independence

Example.

Toss two coins

- $\mathbb{P}(\text{1st } H) =$
- $\mathbb{P}(\text{2nd } H) =$
- $\mathbb{P}(\text{both } H) =$

$$\mathbb{P}(\text{2nd } H \mid \text{1st } H) =$$

Independence

Definition.
Events A and B are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B),$$

otherwise they are *dependent*.

Intuition:

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad \text{when } A,\ B \text{ independent}$$

Properties

Proposition.

If A and B are independent then

- A and B^c are independent,
- A^c and B are independent,
- A^c and B^c are independent.

What about 3 or more events?

Mutual independence

Definition.
A, B and C are *mutually independent* if

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\,\mathbb{P}(B), \\ \mathbb{P}(A \cap C) &= \mathbb{P}(A)\,\mathbb{P}(C), \\ \mathbb{P}(B \cap C) &= \mathbb{P}(B)\,\mathbb{P}(C), \end{aligned}$$

and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C).$$

Mutual independence

Example.

Toss 100 coins.

$$\mathbb{P}(\text{at least 1 head}) =$$

Mutual independence

Remark.

Pairwise independence does NOT imply mutual.

Examples.

1. Toss coins, roll dice, *etc.*
2. Draw two cards *without* replacement
3. Space shuttle. 4 computers, A, B, C, D

$$\mathbb{P}(\text{fail}) = 0.10$$

Space shuttle (cont.)

Scheme: computers in series.

If computers independent,

$$\mathbb{P}(\text{at least one computer works})$$

Bayes' Rule

Theorem.
Let B_1, B_2, ..., B_n be mutually exclusive and exhaustive and let A be an event with $\mathbb{P}(A) > 0$. Then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(B_k)\,\mathbb{P}(A|B_k)}{\sum_{i=1}^{n} \mathbb{P}(B_i)\,\mathbb{P}(A|B_i)}, \quad k = 1, 2, \ldots, n.$$

Bayes' Rule (intuition)

Bayes' Rule: what does it mean?

Given (or know) *a priori* probabilities $\mathbb{P}(B_k)$. Collect some data, which is A.

How to update $\mathbb{P}(B_k)$ to $\mathbb{P}(B_k|A)$?

Example: misfiling assistants

Moe, Larry, and Curly

	Moe	Larry	Curly
Workload	60%	30%	10%

	Moe	Larry	Curly
Prior	$\mathbb{P}(M) =$	$\mathbb{P}(L) =$	$\mathbb{P}(C) =$

Misfiling assistants (cont.)

	Moe	Larry	Curly
Misfile Rate	0.003	0.007	0.010

	Moe	Larry	Curly
Posterior	$\mathbb{P}(M\vert A) \approx$	$\mathbb{P}(L\vert A) \approx$	$\mathbb{P}(C\vert A) \approx$

Random variables

- Experiment E
- Sample space S
- Calculate number X

Definition.

A *random variable* X is a function $X : S \to \mathbb{R}$ that associates to each outcome $\omega \in S$ exactly one number $X(\omega) = x$. The support of X is the set of X's values:

$$S_X = \{x : X(\omega) = x,\ \omega \in S\}$$

Random variables

Example.

Toss a coin three times

Example.

Toss a coin until tails

Example.

Toss a coin, measure time until lands