How to count

Notes

Theorem.
The number of ways to select an *ordered* sample of k subjects from a population that has n distinguishable members is

- n^k if sampling is done with replacement,
- $n(n-1)(n-2)\cdots(n-k+1)$ if sampling is done without replacement.

$n \; n \cdots n$ (k^{th})

$n \; n-1 \; \cdots \; \frac{(n-k+1)}{(n-(k-1))}$

Here, ORDER is IMPORTANT

Examples

Notes

Examples.

1. Flip a coin 7 times $\#(S) = 2^7$
2. 20 students, select president, vice-president, treasurer
   $(20)(19)(18) =$ _____
3. Rent 5 movies. Want to watch 3 movies on the first night.
   $(5)(4)(3)$

How to count

Theorem.
The number of ways to select an *unordered* sample of k subjects from a population that has n distinguishable members is

- $(n-1+k)!/[(n-1)!k!]$ if sampling is done with replacement,
- $n!/[k!(n-k)!]$ if sampling is done without replacement.

$n!/[k!(n-k)!]$ is a *binomial coefficient* "n choose k"

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Notes

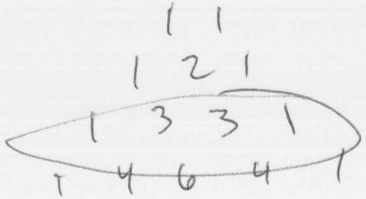
More about binomial coefficients

why called binomial coefficients?

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$



Pascal's Triangle

1 5 10 10 5 1

Notes

Birthday problem

Guess: $\mathbb{P}(\text{match}) = ?$ $< 1\%$? 5%?

A = at least one match, A^c = everybody has diff't b'day.

$\#(S) = 365^n$ $\quad \#(\text{everybody has diff't b'day}) = 365 \cdot 364 \cdots (365-n+1)$

- n people in a class
- 365 days/year, equally likely
- $\mathbb{P}$(at least two have same birthday)

$$1 - \frac{\#(A)}{\#(S)} = 1 - \frac{365(364)\cdots(365-n+1)}{365\,365\cdots(365)}$$

$$= 1 - 1\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\cdots\left(\frac{365-n+1}{365}\right)$$

Notes

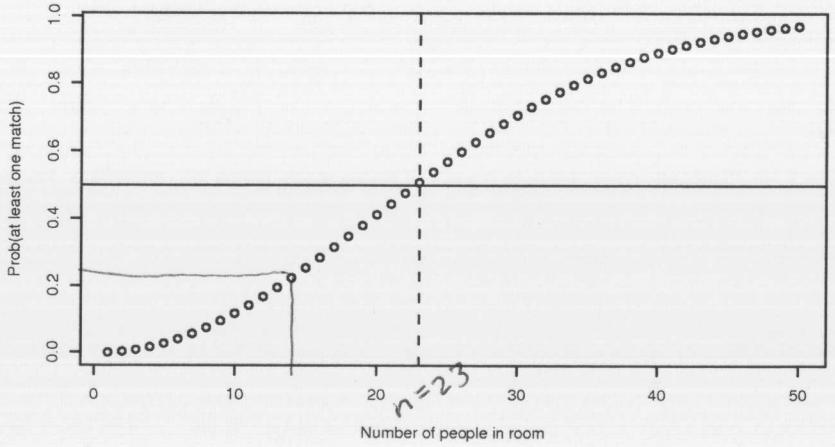


Figure: The birthday problem

Notes

Poker hands

52 cards $\longrightarrow$ 5 card hand

$\#(S) = \binom{52}{5} = 2,598,900$

$S = \{\text{all possible 5 card hands}\}$

(should shuffle 7 times???)

$A = \text{Royal Flush} = \{A, K, Q, J, \text{all same suit}\}$

$\#(A) = 4 \quad \mathbb{P}(A) = \frac{4}{2,598,900} = 0.0000015$

$B = \{\text{Four of a kind}\}$

$\binom{13}{1} = \frac{13!}{1!12!} = \frac{13 \cdot 12 \cdot 11 \cdots}{1 \cdot 12 \cdot 11 \cdots} = 13$

$\#(B) = \binom{13}{1}\binom{4}{4}\binom{48}{1} = (13)(1)(48) = 624$

$\mathbb{P}(B) = \frac{624}{2,598,900} \approx 0.00024$

Notes

$\#(S) = 2,598,900$

$\mathbb{P}(A) = 0.0000015$

$\mathbb{P}(B) = 0.00024$

Conditional probability

52 cards $\longrightarrow$ draw 2 cards (without replacement)

$A = \{\text{1st card drawn is Ace}\}$

$B = \{\text{2nd card drawn is Ace}\}$

Then

$$\mathbb{P}(A) = \frac{4}{52}$$

$$\mathbb{P}(B) = \begin{cases} \frac{3}{51}, & \text{if A happened} \\ \frac{4}{51}, & \text{if A didn't happen} \end{cases}$$

Notes

Conditional probability

Definition.

The conditional probability of B given that the event A occurred is ~~number of ways to select an *unordered* sample of k subjects from a population that has n distinguishable members is~~

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}, \quad \text{if } \mathbb{P}(A) > 0.$$

Notes

Conditional probability

Example

Toss a coin twice.

$S = \{HH, HT, TH, TT\}$

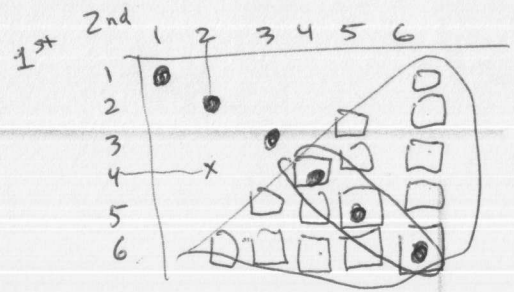
$A = \{\text{a head occurs}\}$ $\quad \mathbb{P}(A) = 3/4$

$B = \{\text{a head and tail occurs}\}$ $\quad \mathbb{P}(B) = 2/4$, $\mathbb{P}(A \cap B) = 2/4$

- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/4}{2/4} = 1$
- $\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{2/4}{3/4} = \frac{2}{3}$

Notes

Conditional probability



Example $\#(S) = 36$

Toss a die twice.

$A = \{\text{outcomes match}\}$

$B = \{\text{sum of outcomes} \geq 8\}$

- $\mathbb{P}(A) = \frac{6}{36} = \frac{1}{6}$
- $\mathbb{P}(B) = 15/36$
- $\mathbb{P}(A \cap B) = 3/36$
- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{3/36}{15/36} = 1/5$
- $\mathbb{P}(B|A) = \mathbb{P}(A \cap B)/\mathbb{P}(A) = \frac{3/36}{6/36} = 1/2$

Notes

Properties

Note.

For any fixed event A with $\mathbb{P}(A) > 0$,

1. $\mathbb{P}(B|A) \geq 0$, for all events $B \subset S$,
2. $\mathbb{P}(S|A) = 1$, and
3. If B_1, B_2, $B_3, \ldots$ are disjoint events, then

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} B_k \,\middle|\, A\right) = \sum_{k=1}^{\infty} \mathbb{P}(B_k|A).$$

Notes

More properties

Note.

For any events A, B, and C with $\mathbb{P}(A) > 0$,

1. $\mathbb{P}(B^c|A) = 1 - \mathbb{P}(B|A)$.
2. If $B \subset C$ then $\mathbb{P}(B|A) \leq \mathbb{P}(C|A)$.
3. $\mathbb{P}[(B \cup C)|A] = \mathbb{P}(B|A) + \mathbb{P}(C|A) - \mathbb{P}[(B \cap C|A)]$.
4. For any two ~~events A and B~~, Multiplication Rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B|A).$$

For 3 events:

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A \cap B)$$

Notes

Conditional probability

Example.

Recall the aces problem

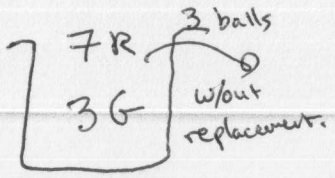
$$A = \{\text{1st card drawn is Ace}\}$$

$$B = \{\text{2nd card drawn is Ace}\}$$

- $\mathbb{P}(\text{both Aces}) = \mathbb{P}(A \cap B)$
  $= \mathbb{P}(A)\mathbb{P}(B|A) = \frac{4}{52} \cdot \left(\frac{3}{51}\right)$

Notes

Conditional probability



Example.

Urn with 10 balls, 7 red and 3 green. Select 3 balls successively from the urn.

$$A = \{\text{1st ball red}\}$$

$$B = \{\text{2nd ball red}\}$$

$$C = \{\text{3rd ball red}\}$$

- $\mathbb{P}(\text{all red}) = \mathbb{P}(A \cap B \cap C)$
  $= \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A \cap B)$
  $= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8}$

Notes

Good example

6 7 7 1

Two urns. First: 6 red, 7 green. Second: 7 red, 1 green
1 ball transferred. Select 1 ball.

$\mathbb{P}(A) = \frac{6}{13}$

$A = \{\text{transferred ball red}\}$

$B = \{\text{selected ball is red}\}$

$\mathbb{P}(B) = B \cap S$
$= B \cap (A \cup A^c)$
$= \mathbb{P}((B \cap A) \cup (B \cap A^c))$
$= \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$
$= \mathbb{P}(A)\mathbb{P}(B|A) + \mathbb{P}(A^c)\mathbb{P}(B|A^c)$
$= \frac{6}{13} \cdot \frac{8}{9} + \frac{7}{13} \cdot \frac{7}{9}$

$\mathbb{P}(B) = \mathbb{P}(\text{red}) = \frac{97}{117}$

Notes

What if you don't look?

Notes

$\mathbb{P}(\text{second card is Ace}) =$

Independence

Notes

Example. $S = \{HH, HT, TH, TT\}$

Toss two coins

- $\mathbb{P}(\text{1st } H) = \frac{2}{4}$
- $\mathbb{P}(\text{2nd } H) = 2/4$
- $\mathbb{P}(\text{both } H) = 1/4$

$$\mathbb{P}(\text{2nd } H \mid \text{1st } H) = \frac{\mathbb{P}(\text{both } H)}{\mathbb{P}(\text{1st } H)}$$

$$= \frac{1/4}{2/4} = 1/2$$

$$= \mathbb{P}(\text{2nd } H)$$

Independence

Definition.
Events A and B are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B),$$

otherwise they are *dependent*.

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B|A)$

Intuition:

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad \text{when } A,\ B \text{ independent}$$

Notes

Properties

Proposition.
If A and B are independent then

- A and B^c are independent,
- A^c and B are independent,
- A^c and B^c are independent.

What about 3 or more events?

A, B, C indep.

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Notes

Mutual independence

Definition.
A, B and C are *mutually independent* if

pairwise independent

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\,\mathbb{P}(B), \\ \mathbb{P}(A \cap C) &= \mathbb{P}(A)\,\mathbb{P}(C), \\ \mathbb{P}(B \cap C) &= \mathbb{P}(B)\,\mathbb{P}(C), \end{aligned}$$

and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C).$$

mutually indep.

n events ?

Notes

Mutual independence

Example.
Toss 100 coins.

$\mathbb{P}(\text{at least 1 head}) =$

$= 1 - \mathbb{P}(\text{all Tails})$

$= 1 - (\mathbb{P}(T) \text{ and } T \text{ and } T \text{ and } \cdots T)$ (100 of these)

$= 1 - (\mathbb{P}(T))(\mathbb{P}(T))\cdots(\mathbb{P}(T))$ (100 of these)

$= 1 - \left(\frac{1}{2}\right)^{100}$

Notes

Mutual independence

Definition.
A, B and C are *mutually independent* if

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\,\mathbb{P}(B), \\ \mathbb{P}(A \cap C) &= \mathbb{P}(A)\,\mathbb{P}(C), \\ \mathbb{P}(B \cap C) &= \mathbb{P}(B)\,\mathbb{P}(C), \end{aligned}$$

and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C).$$

Notes

Mutual independence

Remark.
Pairwise independence does NOT imply mutual.

Examples.

1. Toss coins, roll dice, *etc.* — mutually independent.
2. Draw two cards *without* replacement — Dependent
3. Space shuttle. 4 computers, A, B, C, D

$$\mathbb{P}(\text{fail}) = 0.10$$

If computer A fails, then B activates
B " " C
C " " D

Notes

Space shuttle (cont.)

Notes

Scheme: computers in series.

If computers independent,

$\mathbb{P}$(at least one computer works)

$= 1 - \mathbb{P}(\text{all Fail})$

$= 1 - \mathbb{P}(A \text{ fails and } B \text{ fails and } C \text{ and } D)$

$= 1 - (\mathbb{P}(A \text{ fails})\,\mathbb{P}(B \text{ fails})\,\mathbb{P}(C \text{ fails})\,\mathbb{P}(D \text{ fails})$

$= 1 - \left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)$

$= 1 - \frac{1}{10000} = \frac{9999}{10,000}$

Bayes' Rule

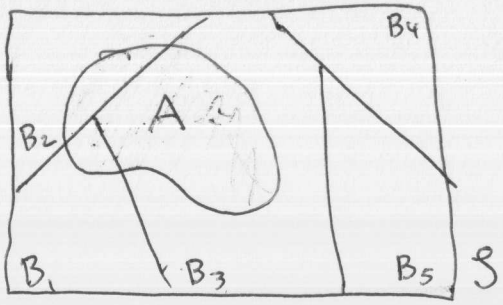
Notes

Theorem.

Let $B_1, B_2, \ldots, B_n$ be mutually exclusive and exhaustive and let A be an event with $\mathbb{P}(A) > 0$. Then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(B_k)\,\mathbb{P}(A|B_k)}{\sum_{i=1}^{n} \mathbb{P}(B_i)\,\mathbb{P}(A|B_i)}, \quad k = 1, 2, \ldots, n.$$

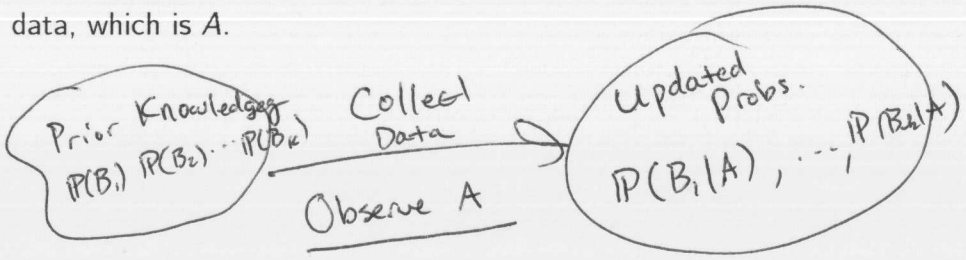
Bayes' Rule (intuition)



Notes

Bayes' Rule: what does it mean?

Given (or know) *a priori* probabilities $\mathbb{P}(B_k)$. Collect some data, which is A.



How to update $\mathbb{P}(B_k)$ to $\mathbb{P}(B_k|A)$?

Notes