Hypothesis Testing

Wish to test hypothesis

$$H_0: p = 0.10 \text{ versus } H_1: p \neq 0.10.$$

How: 1) Go out, collect some data.

2) Assume H_0 is TRUE and construct a $100(1-\alpha)\%$ CI for p.

3) If CI in 2) covers the value $p = 0.10$, then we DO NOT REJECT H_0.

Otherwise, we REJECT H_0.

Remarks:

1) It is possible to be wrong. There are two possible errors.

Type 1 ERROR: Reject H_0 when in fact H_0 is true.

Double Jeopardy, Prison Break, Condemned, Fugitive. Shawshank Redemption...

Type 2 ERROR: Fail to reject H_0 when in fact H_1 is true.

O.J. Simpson. Roethlisberger, Kobe Bryant.

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2) Type I error usually considered worse.

call significance level of the test

$$= \mathbb{P}(\text{Type I error})$$

$$= \alpha.$$ want this to be small. (say $\alpha = 0.05$, $\alpha = 0.01$.)

3) The REJECTION REGION for a test is the set of sample values that lead to rejection of H_0. AKA "critical region" (C.R.).

4) Above test is a "two-sided" or "two-tailed" test.

Can also do one-sided tests.

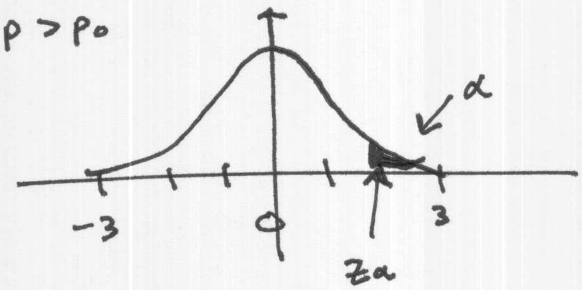
$$H_0: p = p_0 \text{ versus } H_1: p < p_0.$$

Tests of Hypotheses for one Proportion: 11/3/10 (2)

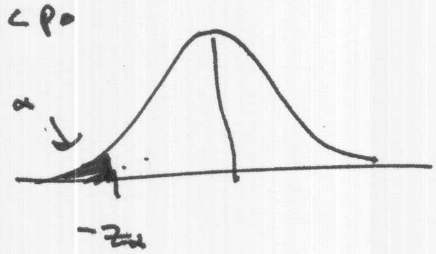
H_0	H_1	Critical Region
$p = p_0$	$p > p_0$	$Z = \dfrac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > z_\alpha$
$p = p_0$	$p < p_0$	$Z < -z_\alpha$
$p = p_0$	$p \neq p_0$	$\lvert Z \rvert > z_{\frac{\alpha}{2}}$

Draw PICTURES

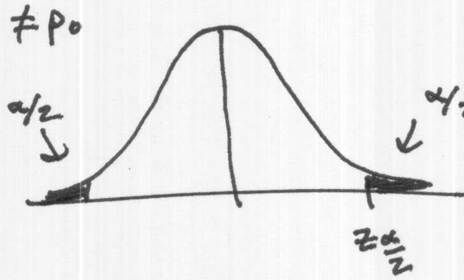
$H_1: p > p_0$



$H_1: p < p_0$



$H_1: p \neq p_0$



Ex) Let p = proportion of defective shotguns. Claim: $p = 0.10$

Install thingamagig, observe $n = 573$ shot guns with 114 defective.

Test Hypotheses

$$H_0: p = 0.10 \quad \text{versus} \quad H: p < 0.10$$

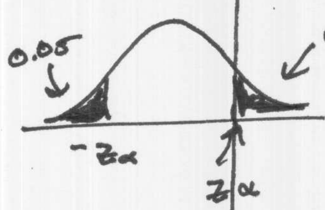
null hyp.   alternative.

Use sig. level $\alpha = 0.05$.

Find critical region.

$$CR = \{ z : z < -z_\alpha \}$$

where $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$ and $\alpha = 0.05$.



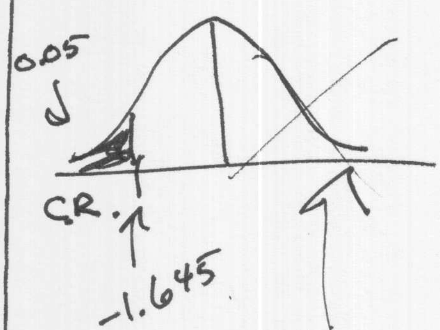
Note:

$$z_\alpha = z_{0.05} =$$

$$= \text{qnorm}(0.05, \text{lower.tail} = \text{FALSE})$$

$$= 1.645$$

$$= \{ z : z < -1.645 \}$$



For our data:

$$z = \frac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$$

$$= \frac{\frac{114}{573} - 0.10}{\sqrt{\dfrac{0.10(1-0.10)}{573}}}$$

$$= 7.89588$$

We FAIL to REJECT H_0.

Conclusion: these data do not provide evidence at the $\alpha = 0.05$ significance level to reject H_0.

$\boxed{\text{Ex}}$ Let p = proportion of dogs who are potty trained.

Suppose hypotheses are

$$H_0 : p = 0.75 \quad \text{and} \quad H_1 : p < 0.75.$$

Collect data and observe $n = 389$ dogs

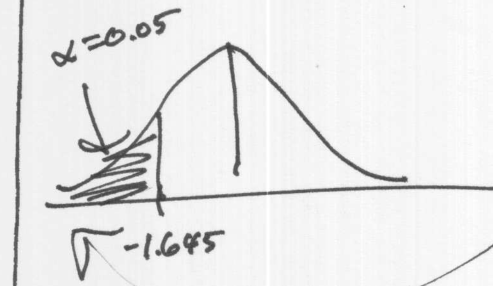
$u = 272$ are potty trained

a) What is your conclusion if $\alpha = 0.05$?

Find Critical Region:

$$CR = \{ z : z < -z_\alpha \}$$
$$= \{ z : z < -z_{0.05} \}$$
$$= \{ z : z < -1.645 \}.$$

where $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} = \dfrac{\frac{272}{389} - 0.75}{\sqrt{\dfrac{0.75(0.25)}{389}}}$



$= -2.312553$

we REJECT H_0.

These data give evidence at $\alpha = 0.05$ level to Reject H_0.

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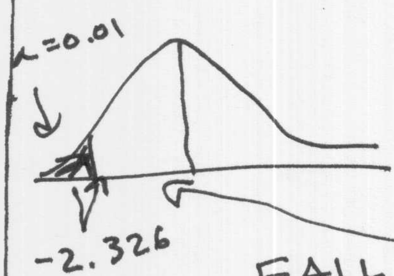
b) What if $\alpha = 0.01$?

Find CR

$$CR : \{ z : z < -z_\alpha \}$$
$$= \{ z : z < -z_{0.01} \}$$
$$= \{ z : z < -2.326 \}$$



Test Statistic

$z = -2.312553$

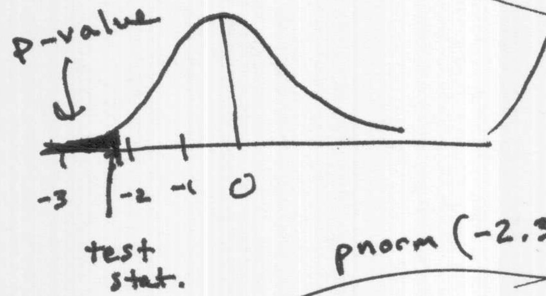
FAIL to REJECT H_0.

The p-value of the test.

p-value = $\mathbb{P}$(obtain the observed value of the test statistic OR MORE EXTREME when H_0 is true)

Find the p-value for our test:

Calculated $z = -2.312553$



pnorm(-2.312553)

$= p$
$= 0.01037361$

Another way to PHRASE hypothesis test: "Reject Ho when p-value $< \alpha$".

Remark: Can do similar thing for two proportions p_1, p_2.

Test of Hypothesis

$$\hat{p}_1 = \frac{y_1}{n_1} \qquad \hat{p}_2 = \frac{y_2}{n_2} \qquad \hat{p} = \frac{y_1 + y_2}{n_1 + n_2}$$

H_0	H_1	Critical Region
$p_1 = p_2$	$p_1 > p_2$	$z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} > z_\alpha$
$p_1 = p_2$	$p_1 < p_2$	$z < -z_\alpha$
$p_1 = p_2$	$p_1 \neq p_2$	$\lvert z \rvert > z_{\frac{\alpha}{2}}$

Tests about one mean and variance

Here $X_1, \ldots, X_n$ are SRS from norm(mean $= \mu$, sd $= \sigma$)

Want to test $H_0 : \mu = \mu_0$.

A: Suppose std. dev. σ is KNOWN.

Then tests are based on

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim \text{norm}(0,1)$$

H_0	H_1	C.R.
$\mu = \mu_0$	$\mu > \mu_0$	$z > z_\alpha$
$\mu = \mu_0$	$\mu < \mu_0$	$z < -z_\alpha$
$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert z \rvert > z_{\frac{\alpha}{2}}$.

B: If σ is unknown:

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t(df = n-1).$$

H_0	H_1	C.R.
$\mu = \mu_0$	$\mu > \mu_0$	$t > t_\alpha(df = n-1)$
$\mu = \mu_0$	$\mu < \mu_0$	$t < -t_\alpha(df = n-1)$
$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert t \rvert > t_{\frac{\alpha}{2}}(df = n-1)$.

Remark: If σ is unknown but n is large then we can use the z-test, . .

Ex) Let X = thickness of quarters in person's pocket.

Data: 1.14, 2.78, . . . 1.268.

Sample size $n = 9$.

Assume $X \sim \text{norm}(\mu, \sigma)$.

Test hypothesis $H_0: \mu = 1$

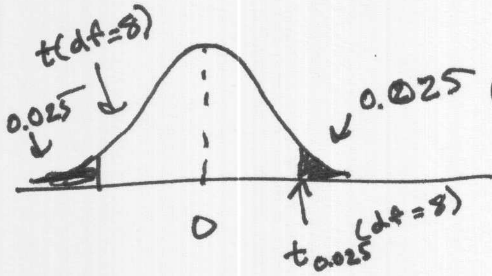
versus $H_1: \mu \neq 1$. Use $\alpha = 0.05$.

Need to use t-test. (small sample, σ unknown).

Need: critical region:

$$CR = \{ t : |t| > t_{\frac{\alpha}{2}}(df = n-1) \}$$

$$= \{ t : |t| > 2.306 \}$$



$qt(0.975) = 2.306004$

Need test statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{\bar{x} - 1}{s/\sqrt{9}}$$

$$\approx 2.103256$$

We fail to reject Ho.

These data do not provide evidence at $\alpha = 0.05$ level.

What's p-value?