5/27/11 (1)

Recall: Bayesian Statistics

$p \to$ prop. of goldfish in lake.

PRIOR: Beta(α, β)

LIKELIHOOD: observed y

POSTERIOR: Beta$(\alpha + y, \beta + n - y)$

Prior Guess at p: $\eta = \frac{\alpha}{\alpha+\beta}$.

NEW GUESS: $\eta^* = \frac{\alpha^*}{\alpha^* + \beta^*}$.

$$= \underbrace{\eta}_{\text{PRIOR MEAN}} \cdot \frac{K}{K+n} + \underbrace{\frac{y}{n}}_{\text{MLE } \hat{p}} \cdot \frac{n}{K+n}$$

POSTERIOR MEAN is a WEIGHTED average of MLE and PRIOR MEAN.

Two things:

as $K \to \infty$, $\eta^* \to \eta$

as $n \to \infty$, $\eta^* \to \frac{y}{n} = $ MLE

REMARKS:

1) How do we choose a Prior?

Notice $\to$ Prior $\to$ Beta dist'n

Posterior $\to$ also Beta.

Beta(α, β) is a CONJUGATE FAMILY.

Beta/Binomial → "Conjugate Pair."

More Conjugate Pairs:

* If $\pi(\theta) = N(\mu, \tau^2)$

  $f(x|\theta) = N(\theta, \sigma^2)$

Then

$$\pi(\theta|x) = N\left(\frac{\bar{x}\tau^2 + \mu\sigma^2/n}{\tau^2 + \sigma^2/n}, \frac{\tau^2 \cdot \sigma^2/n}{\tau^2 + \sigma^2/n} \right)$$

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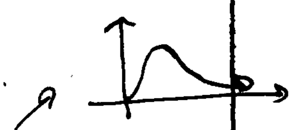
* Gamma/Normal
* Gamma/Poisson
* Gamma/Gamma



Gamma(α, β) has PDF

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0$$

Special Cases:

If $\alpha = 1$ then Gamma$(1, \beta)$ = Exponential(β)

If $\alpha = \nu/2$ and $\beta = 2$

then Gamma$(\nu/2, 2)$

$= \chi^2(\nu)$

"Chi-square with ν degrees of freedom"

Conjugate families chosen for priors historically because

→ EASY
→ TRACTABLE
→ CONVENIENT.

Unfortunately, very RESTRICTIVE.

Used to be a problem. NOT ANYMORE.

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② Bayesian Point Estimation

Our new guess at θ:
Posterior MEAN → $\mathbb{E}[\theta \mid x]$

FACT: $\mathbb{E}(\theta \mid X)$ is OPTIMAL in almost every sense under SQUARED ERROR LOSS.

Def^n: The LOSS of an estimator δ which estimates θ is:

$$L(\delta, \theta) = (\delta - \theta)^2$$

So we will need to compute things like this:

$$\mathbb{E}(\theta \mid x) = \int \theta \cdot \pi(\theta \mid x)\, d\theta$$

$$= \int \theta \cdot \frac{f(x \mid \theta) \cdot \pi(\theta)}{\int f(x \mid \theta) \cdot \pi(\theta)\, d\theta}\, d\theta$$

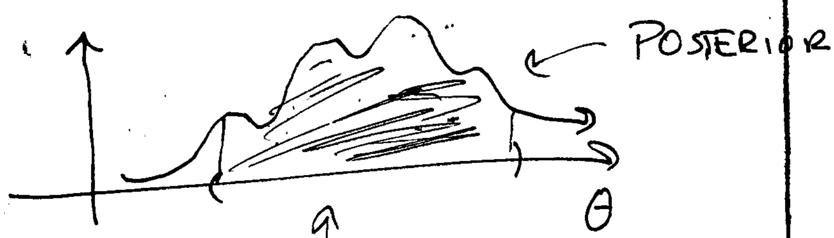
$$= M \cdot \int \theta\, f(x \mid \theta)\, \pi(\theta)\, d\theta$$

This is hard. Need computers.

(3) Bayesian Interval Estimation.

5/27/11 (4)

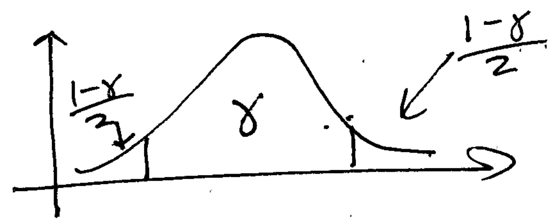
Probability interval of "content" γ.



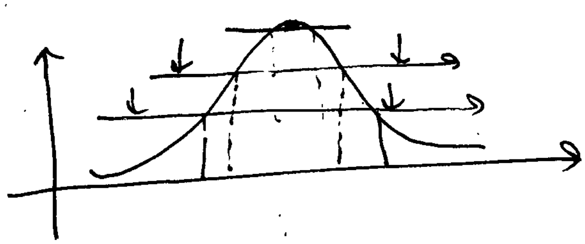
"95% CREDIBLE INTERVAL".

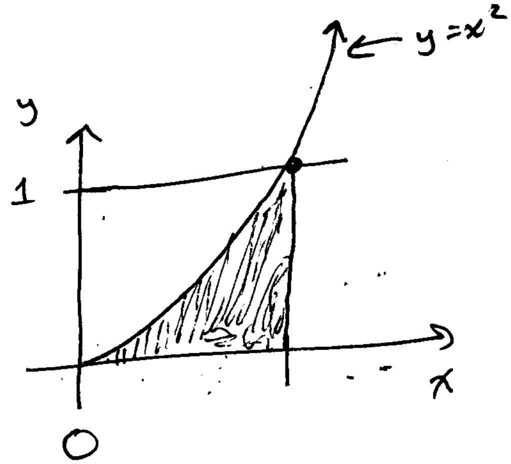
Two examples;

a) EQUAL TAILS interval:



b) Shortest interval of content γ
HDR "Highest Density Region"




y
y = x^2
1
x
0