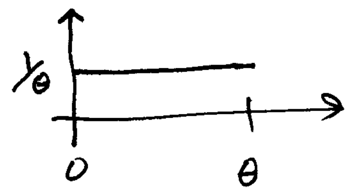
Complete Sufficient Statistics

2/4/11 (1)

T is complete if

$$\mathbb{E}_\theta g(T) = 0 \text{ for all } \theta \text{ implies}$$
$$g \equiv 0 \quad w.p.1.$$

Ex) $X_1, \ldots, X_n$ iid unif$(0,\theta)$



Suff. Stat.

$T = X_{(n)}$
$= \max X_i$.

KNOW: T is minim. suff.

Show T is complete.

Proof: Let g be any function satisfying

$$\mathbb{E}_\theta g(T) = 0 \text{ for all } \theta.$$

Want to show $g \equiv 0$.

$$\mathbb{E}_\theta g(T) = \int_{-\infty}^{\infty} g(t) \cdot \underbrace{f_T(t|\theta)}_{\text{what is this?}} dt$$

$$f_T(t|\theta) = ?$$

$$F_T(t|\theta) = \mathbb{P}_\theta(T \le t)$$
$$= \mathbb{P}_\theta(\max X_i \le t)$$

$$= \mathbb{P}(X_1 \le t, X_2 \le t, \ldots, X_n \le t)$$

$$\overset{\text{indep}}{=} \mathbb{P}(X_1 \le t) \cdot \mathbb{P}(X_2 \le t) \cdots \mathbb{P}(X_n \le t)$$

$$= \left[\mathbb{P}(X_1 \le t)\right]^n$$

$$= \left[F_X(t)\right]^n$$

Therefore

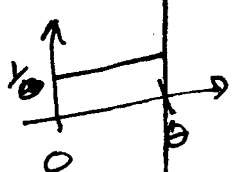
$$f_T(t|\theta) = \frac{d}{dt}\left[F_X(t)\right]^n$$
$$= n\left[F_X(t)\right]^{n-1} \cdot f_X(t)$$

$$\boxed{\begin{array}{l}\underline{\text{Result:}}\ \text{If } X_1, \ldots, X_n \text{ are iid } f_X, \\ \text{then for } T = \max X_i, \\ \quad f_T(t) = n\left[F_X(t)\right]^{n-1} \cdot f_X(t)\end{array}}$$

For unif $(0,\theta)$, $f_X(x) = \frac{1}{\theta}$, $0 < x < \theta$

Therefore

$$F_X(t) = \mathbb{P}(X \le t)$$
$$= \begin{cases} 0 & \text{, if } t \le 0 \\ t/\theta & \text{, if } 0 \le t < \theta \\ 1 & \text{, if } t \ge \theta \end{cases}$$



Thus

$$f_T(t|\theta) = n \cdot \left(\frac{t}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}$$
$$= n\,\theta^{-n} \cdot t^{n-1}, \quad 0 < t < \theta.$$

Back to completeness:

$$\mathbb{E}_\theta\, g(T) = \int_0^\theta g(t) \cdot n\left(\frac{t}{\theta}\right)^{n-1} \frac{1}{\theta}\, dt$$
$$= \int_0^\theta g(t) \cdot n \cdot \theta^{-n} \cdot t^{n-1}\, dt$$

Asssuming

$$n\,\theta^{-n} \int_0^\theta g(t)\, t^{n-1}\, dt = 0 \quad \text{for all } \theta > 0.$$

And so

$$\int_0^\theta g(t)\, t^{n-1}\, dt = 0 \quad \text{for all } \theta > 0.$$

Basic Trick: Take derivative of LHS.

Fund. Thm. of Calc. says

$$\frac{d}{d\theta}\left(\int_0^\theta g(t)\, t^{n-1}\, dt\right) = g(\theta)\cdot \theta^{n-1}, \quad \text{for } \theta > 0$$

Therefore

$$g(\theta)\cdot\theta^{n-1} = 0, \quad \text{for } \theta > 0.$$

$$\Rightarrow \quad g(\theta) = 0, \quad \text{for } \theta > 0.$$

$\therefore$ $T = \max X_i$ is complete. . . Q.E.D.

Ex) $X_1, \dots, X_n$ iid $Norm(\theta, 1)$.

Know: min. suff. stat. is

$$T = \sum_{i=1}^n X_i.$$

Question: is T complete?

Suppose g is a function s.t.

$$E_\theta\, g(T) = 0 \quad \text{for all } \theta.$$

$-\infty < \theta < \infty$.

Want to show:

$$g \equiv 0.$$

$$E_\theta\, g(T) = \int_{-\infty}^{\infty} g(t)\, f_T(t|\theta)\, dt$$

what is this?

KNOW:

$$T \sim Norm(n\theta, n)$$

2/4/11 (4)

$\mathbb{E}_\theta g(T) =$

$$\int_{-\infty}^{\infty} g(t) \frac{1}{\sqrt{n}\sqrt{2\pi}} e^{-\frac{1}{2n}(t-n\theta)^2} dt$$

$$= \frac{1}{\sqrt{2\pi n}} \int_{-\infty}^{\infty} g(t) e^{-\frac{t^2}{2n} + t\theta - \frac{n^2\theta^2}{2n}} dt$$

to be continued...