$X_1, \ldots, X_n \sim \text{Bernoulli}(p)$

4/11/11 (1)

<u>Ex)</u> $H_0: p = 1/4 \qquad (p_0 = 1/4)$

$H_a: p = 3/4 . \qquad (p_1 = 3/4)$

Set $\alpha > 0$ fixed, small.

<u>Testing Function:</u>

$0 \le \phi(x) \le 1$

$$\phi(x) = \begin{cases} 1 , & \text{if } \dfrac{f(x|p_1)}{f(x|p_0)} > k \\ \gamma(x) , & \text{if } \quad '' \quad = k \\ 0 , & \text{if } \quad '' \quad < k . \end{cases}$$

<u>Comments</u>

1) If X is continuous, often $\gamma(x) \equiv 0$.

<u>Reason:</u> $\left\{ x : \dfrac{f(x|\theta_1)}{f(x|\theta_0)} = k \right\}$ has probability zero.

2) If X is discrete, then γ will often be nontrivial, used to achieve an exact significance

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Continue w/ example

Suppose we set $\alpha = 0.05$.

We want

$$\mathbb{E}_{p=1/4} \phi(X) = 0.05 .$$

Here $f(x|p_0) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$

$$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

The ratio

$$\frac{f(x|p=3/4)}{f(x|p=1/4)} = \frac{(3/4)^{\Sigma x_i}(1-3/4)^{n-\Sigma x_i}}{(1/4)^{\Sigma x_i}(1-1/4)^{n-\Sigma x_i}}$$

$$= \left(\frac{3}{\cancel{4}}\right)^{\Sigma x_i} \cdot \left(\frac{\cancel{4}}{1}\right)^{\Sigma x_i} \cdot \left(\frac{1}{\cancel{4}}\right)^{n - \Sigma x_i} \cdot \left(\frac{\cancel{4}}{3}\right)^{n - \Sigma x_i}$$

$$= 3^{2\Sigma x_i - n}$$

$$\doteq 3^{2 \cdot \Sigma x_i - n}$$

This is an increasing function of $T = \sum_{i=1}^{n} X_i$

Hence the MPT can be written in terms of T:

$$\phi(x) = \begin{cases} 1, & \text{if } T > k^* \\ \gamma^*(x), & \text{if } T = k^* \\ 0, & \text{if } T < k^* \end{cases}$$

To figure out k^* and γ^*:

Want to find the dist'n of T.

$$T \sim \text{binomial}(n, p).$$

Take, for example, $\boxed{n = 10}$

We want to find k^* such that when $p = 1/4$, the probability of reject is as close as possible to $\alpha = 0.05$.

Take a look at function

$$H(k) = \mathbb{P}_{p=1/4}(T > k).$$

Make Table:

4/11/11 (3)

k	$\mathbb{P}(T > k)$ when $p = 0.25$
0	0.944
1	0.756
2	0.474
3	0.224
4	0.078
$\boxed{5}$	$\boxed{0.020}$
6	0.004
7	0.
8	
⋮	

closest to $\alpha = 0.05$ without going over.

to 3 decimal spots.

Choose $\underline{k^* = 5.}$

Then
$$\mathbb{E}_{p=1/4}\,\phi(X) = 1 \cdot \mathbb{P}_{p=1/4}(T > k^*) + \gamma(x) \cdot \mathbb{P}_{p=1/4}(T = k^*) + \cancel{0 \cdot \mathbb{P}_{p=1/4}(T < k^*)}$$

$$= 0.020 + \gamma(x) \cdot \underbrace{\mathbb{P}_{p=1/4}(T = k^*)}_{\text{this is } 0.058}$$

Want this to equal α

$$\underbrace{0.050} = 0.020 + \gamma(x) \cdot 0.058.$$

$$0.030 = 0.058 \cdot \gamma$$

$$0.517 \approx \frac{0.030}{0.058} = \gamma(x)$$

Ultimate Conclusion: The MPT of H_0 versus H_a that has level $\alpha = 0.05$ is

$$\phi(x) = \begin{cases} 1, & \text{if } \sum X_i > 5, \\ 0.517, & \text{if } \sum X_i = 5, \\ 0, & \text{if } \sum X_i < 5. \end{cases}$$

Remarks:

1) The MPT is NOT UNIQUE!

a) $$\phi = \begin{cases} 1 & \text{if } T > 5 \\ 0.517, & \text{if } T = 5 \\ 0, & \text{if } T < 5 \end{cases}$$

b) Another MPT:

$$\gamma^{**}(x) = \begin{cases} \frac{0.517}{2}, & \text{if } \begin{array}{l} x_1 = x_2 = \cdots = x_5 = 1 \\ x_6 = \cdots = x_{10} = 0 \end{array} \\ \frac{0.517}{2}, & \text{if } \begin{array}{l} x_1 = \cdots = x_5 = 0 \\ x_6 = \cdots = x_{10} = 1 \end{array} \\ 0, & \text{otherwise} \end{cases}$$

Our new MPT looks like

$$\phi^{**} = \begin{cases} 1, & \text{if } T > 5 \\ \gamma^{**}, & \text{if } T = 5 \\ 0, & \text{otherwise.} \end{cases}$$

Another Example:

Let $X \sim f(x) = e^{-x}$, $x > 0$.

Observe $Y = X^{\theta}$.

Find the PDF of Y.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) \\ &= \mathbb{P}(X^{\theta} \le y) \\ &= \mathbb{P}((X^{\theta})^{1/\theta} \le y^{1/\theta}) \\ &= \mathbb{P}(X \le y^{1/\theta}) \\ &= F_X(y^{1/\theta}). \end{aligned}$$

and BTW, $F_X(x) = 1 - e^{-x}$.

$$= 1 - e^{-y^{1/\theta}}.$$

Therefore,

$$f_Y(y) = \left(F_Y(y)\right)'$$
$$= \left(1 - e^{-y^{1/\theta}}\right)'$$
$$= -e^{-y^{1/\theta}} \cdot \left(-\frac{1}{\theta} y^{1/\theta - 1}\right)$$
$$= \frac{1}{\theta} y^{1/\theta - 1} e^{-y^{1/\theta}}, \quad \text{for } y > 0.$$