Pivotal Quantities

5/2/11 (1)

Remark: $T \equiv 1$ is definitely a pivotal quantity.

Let $X_1, \ldots, X_n \sim f(x|\theta)$.

Let S be a sufficient statistic.

Let $S \sim$ CDF $F_\theta(s)$.

Suppose F_θ is continuous.

Let
$$\begin{aligned} U(X, \theta) &= U \\ &= F_\theta(S) \\ &\sim \text{UNIFORM}(0,1) \end{aligned}$$

does ~~not depend~~ on θ.

U is a pivotal quantity.

$$\Rightarrow A = \left\{ \theta : \frac{\alpha}{2} \le U(X, \theta) \le 1 - \frac{\alpha}{2} \right\}$$

is a $100(1-\alpha)\%$ confidence interval for θ.

Ex) $X_1, \ldots, X_n \sim N(\theta, 1)$.

Then $S = \bar{X}$. KNOW

$$S \sim N(\theta, \tfrac{1}{n}).$$

$$F_\theta(s) = \Phi(\sqrt{n}(s - \theta)).$$

That means

$$U = F_\theta(S) = \Phi(\sqrt{n}(\bar{X} - \theta))$$

$$A = \left\{ \theta : \frac{\alpha}{2} \le \Phi(\sqrt{n}(\bar{x} - \theta)) \le 1 - \frac{\alpha}{2} \right\}$$

is a $100(1-\alpha)\%$ CI for θ.

Rewrite A:

$$= \left\{ \theta : -z_{\frac{\alpha}{2}} \le \sqrt{n}(\bar{X} - \theta) \le z_{\frac{\alpha}{2}} \right\}$$

$$= \left\{ \theta : -\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \le \bar{X} - \theta \le z_{\alpha/2} \cdot \frac{1}{\sqrt{n}} \right\}$$

$$= \left\{ \theta : -\bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}} \le -\theta \le -\bar{X} + z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}} \right\}$$

$$= \left\{ \theta : \bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}} \le \theta \le \bar{X} + z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}} \right\}$$

$$\bar{X} \pm z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}}$$

One-sided Intervals

$X \sim f(x|\theta)$

$L(X)$ = statistic, called a LOWER BOUND for θ with confidence level $100(1-\alpha)\%$ if

$$\mathbb{P}_\theta(\theta \ge L(\underline{X})) \ge 1 - \alpha$$

and $U(\underline{X})$ is an UPPER BOUND if

$$\mathbb{P}_\theta(\theta \le U(\underline{X})) \ge 1 - \alpha, \text{ for all } \theta.$$

How to find them:

① Inverting a test (Lower Bound)

Test hypotheses

$H_0: \theta \ge \theta_0$ versus

$H_a: \theta < \theta_0.$

Write $A(\theta)$ = acceptance region for test, if we are lucky, the region looks like

$$[L(x), +\infty).$$

For an UPPER Bound, instead test $H_0: \theta \le \theta_0$ versus $H_a: \theta > \theta_0$.

Method

(2) Pivotal Quantities

$Q(x, \theta)$ is pivotal.

Assume $Q(x, \theta)$ is increasing in θ for each x.

Then set

$$\{\theta : Q(x,\theta) \ge q_\alpha\}$$

$$=: \{\theta : \theta \ge \underbrace{Q^{-1}(q_\alpha, x)}_{\text{This is } L(X),}\}$$

Example: $X_1, \ldots, X_n$ IID UNIFORM $(0, \theta)$.

Want Lower Bound for θ.

1) Inverting a test.

$H_0: \theta \le \theta_0$ versus. $H_a: \theta > \theta_0$. Fixed $\theta_0 > 0$.

Then the UMP test Rejects H_0 if

$$X_{(n)} > c.$$

where c is such that

$$\mathbb{P}_{\theta_0}(X_{(n)} > c) = \alpha.$$

Let $T = \dfrac{X_{(n)}}{\theta_0} = \max\left(\dfrac{X_i}{\theta_0}\right)$

$= \max(U_i)$ when H_0 is true.

U_i: uniform $(0,1)$.

Then

$T \sim$ pdf:

$$f(t) = n \cdot t^{n-1}, \quad 0<t<1.$$

Therefore

$$\mathbb{P}_{\theta_0}\{X_{(n)} > c\} = \mathbb{P}_{\theta_0}\left(\frac{X_{(n)}}{\theta_0} > \frac{c}{\theta_0}\right)$$

$$= \mathbb{P}\left(T \geq \frac{c}{\theta_0}\right)$$

$$= n\int_{c/\theta_0}^{1} t^{n-1}\,dt$$

$$= 1 - \left(\frac{c}{\theta_0}\right)^n.$$ Set this equal to α

$$\alpha = 1 - \left(\frac{c}{\theta_0}\right)^n$$

$$\left(\frac{c}{\theta_0}\right)^n = 1-\alpha$$

$$\frac{c}{\theta_0} = (1-\alpha)^{1/n}$$

$$\boxed{c = \theta_0(1-\alpha)^{1/n}}$$

So our Acceptance Region is

$$A(\theta_0) = \{\theta : X_{(n)} \leq \theta_0(1-\alpha)^{1/n}\}$$

Bound $= \{\theta : x \in A(\theta)\}$

$= \{\theta : X_{(n)} \leq \theta(1-\alpha)^{1/n}\}$

$= \boxed{\{\theta : \dfrac{X_{(n)}}{(1-\alpha)^{1/n}} \leq \theta\}}$ 100(1-α)% CI that gives a lower bound for θ.

Method 2: Pivotal Quantity – based on $Q = \frac{\max X_i}{\theta}$.

Have seen how to find CIs.

Question: how to ASSESS CI's?

That is, suppose have 2 intervals

$I_1 = [L_1, U_1]$, $I_2 = [L_2, U_2]$, with $\mathbb{P}_\theta(\theta \in I_i) = 1-\alpha$, $i = 1, 2$.

Which one's better?

One criterion: SKINNIER is BETTER.

$[0.41, 0.48]$

$\cancel{[0.42, 0.69]} \rightarrow$ WIDER INTERVAL

We say I_1 is better than I_2 if

$$|I_1| \le |I_2|$$

or $$\mathbb{E}|I_1| \le \mathbb{E}|I_2|.$$

Ex) $X_1, \ldots, X_n$ iid $N(0, \sigma^2)$.

Conf. Int for σ^2.

Based on Pivotal Quantity

$$\frac{\sum X_i^2}{\sigma^2} \sim \chi^2(df = n).$$

So our interval looks like

$$\left\{ \sigma^2 : a \le \frac{\sum X_i^2}{\sigma^2} \le b \right\}$$

or $$\boxed{\left\{ \sigma^2 : \frac{\sum x_i^2}{b} \le \sigma^2 \le \frac{\sum x_i^2}{a} \right\}}$$