Method 1

Given $X_1, \ldots, X_n$ iid $N(0, \sigma^2)$.
Want CI for σ^2.
Use pivotal quantity

$$\frac{\sum X_i^2}{\sigma^2} \sim \chi^2(df = n) = \chi_n^2.$$

This gives us interval that looks like

$\left\{ a < \frac{\sum X_i^2}{\sigma^2} < b \right\}$

$$\left\{ \sigma^2 : \frac{\sum X_i^2}{b} \le \sigma^2 \le \frac{\sum X_i^2}{a} \right\}.$$

that is,

$$\left[\frac{\sum X_i^2}{b}, \frac{\sum X_i^2}{a} \right],$$ where

a and b are such that

$$\int_a^b \chi_n^2(x)\,dx = \int_a^b \underbrace{\frac{1}{\Gamma(\frac{n}{2}) 2^{n/2}} x^{n/2 - 1} e^{-x/2}}_{\chi_n^2(x)}\,dx$$

$$= 1 - \alpha.$$

The Length of the interval is

$$L = \frac{\sum X_i^2}{a} - \frac{\sum X_i^2}{b}$$

so the expected length is

$$\mathbb{E}_{\sigma^2} L = \mathbb{E}\left(\frac{\sum X_i^2}{a} - \frac{\sum X_i^2}{b} \right)$$

$$= \left(\frac{1}{a} - \frac{1}{b} \right) \mathbb{E}\left(\sum X_i^2 \right)$$

$$= n\sigma^2 \left(\frac{1}{a} - \frac{1}{b} \right)$$

5/4/11 (2)

Question: What choices of a, b result in intervals with small expected length? Is there a "best" choice?

OPTIMIZATION PROBLEM

Find constants a, b such that

1) $$\int_a^b \frac{1}{\Gamma(\frac{n}{2})2^{n/2}} x^{n/2-1} e^{-x/2}\,dx = 1-\alpha$$

2) $$n\left(\frac{1}{a}-\frac{1}{b}\right)$$ is <u>minimum.</u>

Know that b is a function of a, that is, $b = b(a)$.

Let $h(a) = n\left(\frac{1}{a} - \frac{1}{b(a)}\right)$.

Get derivative.

$$h'(a) = n\left(-\frac{1}{a^2} + \frac{1}{b^2(a)}\cdot b'(a)\right)$$

So $h' = 0$ when

$$n\left(-\frac{1}{a^2} + \frac{b'}{b^2}\right) = 0$$

$$-\frac{1}{a^2} + \frac{b'}{b^2} = 0$$

$$\frac{b'}{b^2} = \frac{1}{a^2}$$

$$\boxed{b' = \frac{b^2}{a^2}} \quad *$$

Take the derivative of 1:

$$\frac{d}{da}\int_a^{b(a)} \chi^2_n(x)\,dx = \frac{d}{da}(1-\alpha)$$

$$\frac{d}{da}\left(\int_0^{b(a)} \chi^2(x)\,dx - \int_0^a \chi^2(x)\,dx\right) = 0.$$

$$\chi^2_n(b)\cdot b' - \chi^2_n(a) = 0$$

$$\chi^2_n(b)\cdot b' = \chi^2_n(a)$$

$$b' = \frac{\chi^2_n(a)}{\chi^2_n(b)}$$

$$= \frac{\cancel{\frac{1}{\Gamma(n/2)}\frac{1}{2^{n/2}}}\cdot a^{n/2-1}e^{-a/2}}{\cancel{\frac{1}{\Gamma(n/2)}\frac{1}{2^{n/2}}}\cdot b^{n/2-1}e^{-b/2}}$$

$$\boxed{b' = \frac{a^{n/2-1}e^{-a/2}}{b^{n/2-1}e^{-b/2}}} \qquad **$$

Since $(*) = (**)$ we get

$$\frac{b^2}{a^2} = \frac{a^{n/2-1}e^{-a/2}}{b^{n/2-1}e^{-b/2}}$$

or

$$\boxed{b^{n/2+1}e^{-b/2} = a^{n/2+1}e^{-a/2}}$$

Conclusion: The choice of (a,b) which minimizes expected length satisfy the equations

① $\int_a^b \chi^2_n(x)\,dx = 1-\alpha$

② $b^{n/2+1}e^{-b/2} = a^{n/2+1}e^{-a/2}$

Method 2: Invert a hypothesis test.

Start w/ hypotheses

$$H_0: \sigma^2 = \sigma_0^2 \text{ versus } H_a: \sigma^2 \neq \sigma_0^2.$$

Need Likelihood function

$$L(\sigma^2) = c\,\sigma^{-n} e^{-\frac{\sum x_i^2}{2\sigma^2}}.$$

<u>LRT</u>: restricted MLE $\widehat{\sigma_0^2} = \sigma_0^2$

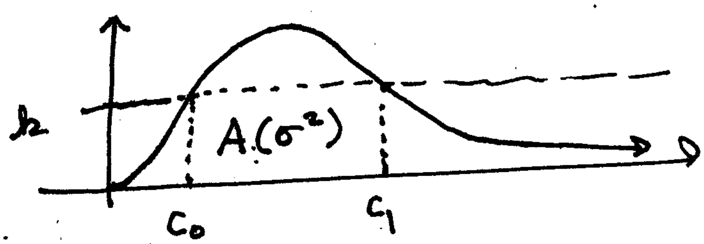
unrestricted MLE $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$$\lambda = \frac{L(\widehat{\sigma_0^2})}{L(\widehat{\sigma^2})} = c^*\left(\frac{\sum x_i^2}{\sigma_0^2}\right)^n e^{-\frac{\sum x_i^2}{2\sigma_0^2}}$$

Let $h(x) = c^* x^n \cdot e^{-x/2}$

Then LRT $\lambda = h\left(\frac{\sum x_i^2}{\sigma_0^2}\right)$.

Reject H_0 if $\lambda \le k$



So the interval based on the acceptance region looks like

$$\left\{\sigma^2 : c_0 \le \frac{\sum x_i^2}{\sigma^2} \le c_1\right\},$$ where

c_0 and c_1 satisfy

① $\int_{c_0}^{c_1} \chi_n^2(x)\,dx = 1 - \alpha$, and

② $h(c_0) = h(c_1)$ (equal function height).

That is, $h(x) = c^* x^n e^{-x/2}$

$$h(c_0) = h(c_1)$$

$$\cancel{c^*} c_0^n e^{-c_0/2} = \cancel{c^*} c_1^n e^{-c_1/2}$$

$$c_0^n e^{-c_0/2} = c_1^n e^{-c_1/2}.$$

In conclusion; The interval based on inverting the LRT looks like $\left\{\sigma^2 : \frac{\sum X_i^2}{c_1} \le \sigma^2 \le \frac{\sum X_i^2}{c_0}\right\}$

where c_0 and c_1 satisfy

1. $\int_{c_0}^{c_1} \chi_n^2(x)\,dx = 1-\alpha$
2. $c_0^n e^{-c_0/2} = c_1^n e^{-c_1/2}$

Ultimate Conclusion: the interval based on LRT is different from the one we got from optimizing expected length with pivotal quantity.

Morals of story;

1. LRT interval isn't necessarily best w.r.t. expected length.

Remark: even though the LRT interval didn't do well this time, under general assumptions it is optimal w.r.t. expected length.