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Chapter 3: TRENDS. → Deterministic (Regression).

Chapter: Models for Stationary Time Series.

General Linear Process

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

We also assume

$$\sum_{k=1}^{\infty} \psi_k^2 < \infty$$

can assume $\psi_0 = 1$.

Example: Suppose $\psi_k = \phi^k$

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} \cdots$$

$$E Y_t = E\left(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots\right)$$

$$= 0 + \phi \cdot 0 + \phi^2 \cdot 0 + \cdots$$
$$= 0.$$

Variance: γ_0

$$Var\left(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots\right)$$

$$= Var(e_t) + Var(\phi e_{t-1}) + Var(\phi^2 e_{t-2}) + \cdots$$
$$= \sigma^2 + \phi^2 \cdot \sigma^2 + \phi^4 \cdot \sigma^2 + \cdots$$
$$= \sigma^2 (1 + \phi^2 + \phi^4 + \cdots)$$
$$= \sigma^2 \cdot \sum_{k=0}^{\infty} [\phi^2]^k$$

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$$= \sigma^2 \cdot \frac{1}{1-\phi^2}, \quad |\phi^2| < 1$$
$$-1 < \phi < 1.$$

$$\gamma_1 = Cov(Y_t, Y_{t-1})$$
$$= Cov(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots, e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \cdots)$$
$$= 0 + \phi\sigma^2 + \phi^3\sigma^2 + \phi^5\sigma^2 + \phi^7\sigma^2 \cdots$$
$$= \phi\sigma^2(1 + \phi^2 + \phi^4 + \cdots)$$
$$= \frac{\phi\sigma^2}{1-\phi^2}$$

Then autocorrelation:

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi\cancel{\sigma^2/1-\phi^2}}{\cancel{\sigma^2/1-\phi^2}}$$
$$= \phi$$

Similarly, $\boxed{\rho_k = \phi^k}$

Does NOT Depend on t! STATIONARY!

For general linear:

$$\gamma_k = \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$$

Moving Average Process:

only finitely many of ψ's non-nonzero.

Example: MA(1)

$$Y_t = e_t + \theta e_{t-1}, \quad t = 1, 2, 3, \ldots$$

Then

$$\mathbb{E}\, Y_t = \mathbb{E}(e_t + \theta e_{t-1})$$
$$= 0 + 0 = 0.$$

$$\text{Var}(Y_t) = \text{Var}(e_t + \theta e_{t-1})$$
$$= \sigma^2 + \theta^2 \cdot \sigma^2$$
$$= \sigma^2(1+\theta^2)$$
$$= \gamma_0$$

$$\gamma_1 = \theta \cdot \sigma^2$$

which means

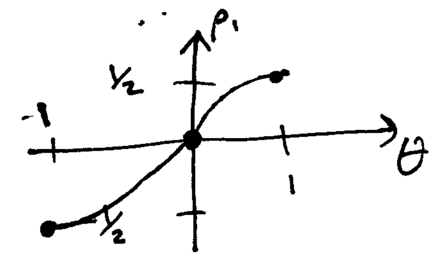
$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta \cancel{\sigma^2}}{\cancel{\sigma^2}(1+\theta^2)} = \theta/(1+\theta^2)$$

$$\rho_1 = \frac{\theta}{1+\theta^2}$$

and

$\rho_k = 0$ otherwise...

How does ρ change with θ?



WATCH THIS:

$$\rho = \frac{\theta}{1+\theta^2}$$

What if plug in $\frac{1}{\theta}$?

$$\frac{(1/\theta)}{1+(1/\theta)^2} = \frac{1/\theta}{1+1/\theta^2}\frac{\theta^2}{\theta^2} = \frac{\theta}{\theta^2+1}$$

That is,

MA(1) with $\theta = 0.7$

has same autocorrelation as

MA(1) with $\theta = 10/7$

Invertibility

Ex) Plots.

Next: MA(2)

$$Y_t = e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}$$

$$\text{Mean} = E(e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2})$$
$$0 + 0 + 0$$
$$= 0$$

$$\text{Var}(Y_t) = V(e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2})$$
$$= \sigma^2 + \theta_1^2 \cdot \sigma_2 + \theta_2^2 \sigma^2$$
$$= \sigma^2 (1 + \theta_1^2 + \theta_2^2)$$

$$\gamma_1 = \theta_1 + \theta_1 \theta_2 \sigma^2$$
$$\gamma_2 = \theta_2 \sigma^2$$
$$\gamma_3 = 0 \ldots$$

And so autocorrelation

$$\rho_1 = \frac{\gamma_1}{\gamma_0}, \quad \rho_2 = \frac{\gamma_2}{\gamma_0}, \quad \rho_k = 0 \text{ otherwise}$$

General MA(q) process

$$Y_t = e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$$

$$\text{Var}(Y_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)$$

Auto correlation:

$$\rho_k = \frac{\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2}, \quad k=1,\ldots,q$$

and ρ_q has numerator θ_q.

and $\rho_k = 0$ otherwise.

"Cuts off"

Autoregressive Process: regression on itself.

- depends on past values of series plus "innovation"

$\boxed{\text{Ex}}$ AR(1)

$$Y_t = \phi Y_{t-1} + e_t$$

Mean: $E(Y_t) = E(\phi Y_{t-1} + e_t)$

☺ = ϕ☺ + 0

Variance: $\text{Var}(Y_t) = \text{Var}(\phi Y_{t-1} + e_t)$

$$\sigma_y^2 = \phi^2\sigma_y^2 + \sigma_e^2$$

$$\sigma_y^2 - \phi^2\sigma_y^2 = \sigma_e^2$$

$$\sigma_y^2(1-\phi^2) = \sigma_e^2$$

$$\sigma_y^2 = \frac{\sigma_e^2}{1-\phi^2}$$

Autocovariance:

$$Y_{t-k} Y_t = \phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t$$

$$E(\quad) = E(\quad) + \cancel{E(\quad)}$$

$$\Downarrow$$

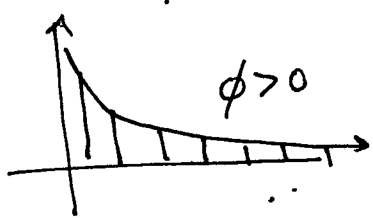
$$\gamma_k = \phi \gamma_{k-1} + 0$$

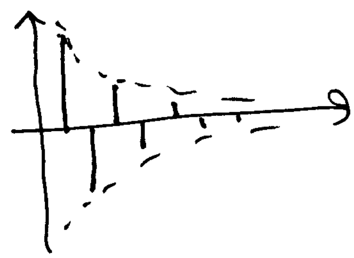
Thus $\gamma_k = \phi^k \cdot \dfrac{\sigma^2}{1-\phi^2}$, $k=1,\dots$

And $\rho_k = \phi^k$, $k=1,2,3,\dots$

STATIONARY.

Make plots of ACF





Stationarity condition: need

$$|\phi| < 1.$$

General Linear process:

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\
&= \phi(\phi(\phi Y_{t-3} + e_{t-2}) + e_{t-1}) + e_t \\
&\;\vdots \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots
\end{aligned}$$

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Next: AR(2)

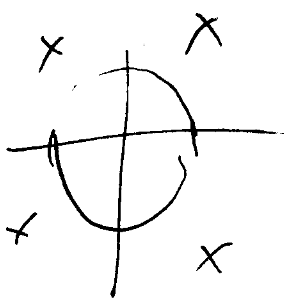
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

"Characteristic Polynomial"

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2$$

"Characteristic Equation"

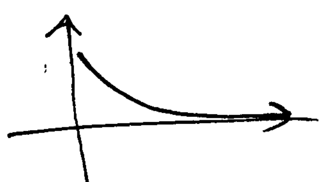
$$\phi(x) = 0$$



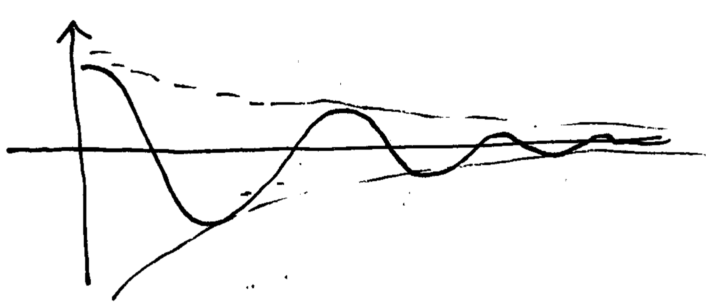
- Stationarity condition:
  Need roots of CE to be greater than 1 in modulus.

- ACF: complicated. Depends on roots of CE.

Shape: If real roots, ACF goes to zero exponentially



If roots of CE are complex, looks like "damped sine function"



Yule-Walkers Equations:

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$$

$k = 2, 3, 4, \ldots$

General Linear process
MORE COMPLICATED,
depends on roots of CE.

General AR(p)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Characteristic poly:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 \cdots - \phi_p x^p.$$

Char. Eqt^n:

$$\phi(x) = 0.$$

All roots must be greater than 1 in modulus to be stationary AR(p)

A.K.A. "causal model".

Necessary conditions.

$$\sum_{k=1}^{p} \phi_k < 1 \quad \text{and} \quad |\phi_p| < 1$$

Mixed Autoregressive Moving Average Models "ARMA(p,q)"

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t + \cdots$$

$$\cdots + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$$

Stationarity: need roots of AR polynomial fall outside unit circle.

Also need: no common factors in AR M.A. characteristic polynomial.

AR model: $\phi(x) = 1 - \phi_1 x - \cdots - \phi_p x^p$.

MA charact. polynomial

$$\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q .$$

Example: $Y_t = e_t$ White noise $*$

$$Y_{t-1} = e_{t-1}$$

$$0.7\, Y_{t-1} = 0.7\, e_{t-1} \quad **$$

Then what is

$* - **$

$$Y_t - 0.7 Y_{t-1} = e_t - 0.7\, e_{t-1}$$

$$Y_t = 0.7 Y_{t-1} + e_t - 0.7 e_{t-1}$$

this looks like

ARMA(1, 1)!!

But look at Char. polynomials:

AR: $\phi(x) = 1 - 0.7x$

MA: $\theta(x) = 1 - 0.7x$

SAME!

Example: ARMA(1,1)

$$Y_t = \phi Y_{t-1} + e_t + \theta e_{t-1}$$

Always assume no common factors for polyn. which means $\phi \neq \theta$

Variance:

$$\gamma_0 = \sigma_e^2 (1 + 2\phi\theta + \theta^2) / (1 - \phi^2)$$

Autocorrelation:

$$\rho_k = \frac{\phi^{k-1}(1 + \theta\phi)(\phi + \theta)}{1 + 2\theta\phi + \theta^2}$$

General linear process

$$Y_t = e_t + (\phi + \theta) \cdot \sum_{j=1}^{\infty} \phi^{j-1} e_{t-j}$$

Fact: MA models can be nonunique in general.

KNOW: Can rewrite AR process as linear process.

Ex) KNOW AR(1)

$$Y_t = e_t + \underset{\substack{\downarrow \\ \theta_1}}{\phi} e_{t-1} + \underset{\substack{\downarrow \\ \theta_2}}{\phi^2} e_{t-2} + \cdots$$

Conclusion: AR(p) is just an MA(∞) process!

Question: Can you start with MA(q), jumble around, get

AR(∞) ????

Yes!!! (sometimes).

Look at MA(q) characteristic polynomial

$$\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q .$$

We say an MA(q) model is "INVERTIBLE" iff all roots of $\theta(x) = 0$ to fall ~~outside unit~~ circle.

Invertible models are UNIQUE for given autocorrelation

Exercises:

Undergrad 2-5, 7-10

Grad: 12, 15, 18, 23

Note: check out

ARMAacf function in TSA package.