Rectilinear Motion

An interest in methods to study the nature of continuous motion was one of the principal reasons for the development of calculus. In this section we use the derivative to describe the motion of a particle that moves in a straight line. This is called **rectilinear motion**.

Suppose a rock is dropped from a height of 1600 ft and that the height, in feet, of the rock above the ground t sec after it has been dropped is given by

$$s(t) = 1600 - 16t^2, \quad \text{where } 0 \leq t \leq 10$$

How fast is the rock moving 5 sec after it has been dropped?

The *average* rate at which an object travels is found by dividing the distance traveled by the time required to travel that distance.

The **Average velocity** describes this rate as well as the direction in which the moving.

The **average speed** of an object is the magnitude, or absolute value, of its average velocity.

For the falling rock in our example, the distance traveled is the difference between the height of the rock at two different times. Because the rock is falling to earth, the distance from the ground is decreasing with time, and the average velocity is always negative. For example, the average velocity of the rock from $t = 5$ sec to $t = 6$ sec is

$$\text{Average velocity} = \frac{s(6) - s(5)}{(6-5)}\ \frac{\text{ft}}{\text{sec}} = (1024 - 1200)\ \frac{\text{ft}}{\text{sec}} = -176\ \frac{\text{ft}}{\text{sec}}.$$

This calculation gives the *average* velocity of the rock for the 1-sec interval from $t = 5$ sec to $t = 6$ sec. The question "How fast is the moving 5 sec after it has been dropped?" is asking for the *instantaneous* velocity at $t = 5$ sec. This velocity, denoted $v(t)$, is the instantaneous rate of change of $s(t)$ with respect to the change in t.

In Chapter 2 we found that the *derivative* of a function describes the instantaneous rate of change in the values of a function with respect to change in its independent variable. Thus the velocity of the rock, in feet per second, at $t = 5$ sec is

$$v(5) = s'(5) = \lim_{h \to 0} \frac{s(5+h) - s(5)}{h}.$$

Since $s(t) = 1600 - 16t^2$,

$$v(t) = s'(t) = -32t, \quad \text{and} \quad v(5) = -160\ \frac{\text{ft}}{\text{sec}}.$$

The negative value for $v(t)$ implies that $s(t)$ is decreasing with time. This is, of course, the case since the rock is falling.

The rate at which the velocity of an object changes with respect to t is also of interest in physical problems. The derivative of the velocity gives this instantaneous rate of change, which is called the *acceleration* and denoted $a(t)$. For the falling rock, $v(t) = -32t$ ft/sec, so

$$a(t) = v'(t) = s''(t) = -32\ \frac{\text{ft/sec}}{\text{sec}} = -32\ \frac{\text{ft}}{\text{sec}^2}.$$

This connection between the derivative and the *motion equations* of objects is one of the fundamental discoveries in the history of science and is the beginning of the study of physics as we know it. It permits us to the following definition.

Suppose $s(t)$ describes the rectilinear (straight-line) motion of an object.

- The **velocity** of the object at time t, $v(t)$, is defined by $v(t) = s'(t)$, provided $s'(t)$ exists.
- The **speed** of the object is defined to be the magnitude of the velocity, $|v(t)|$.
- The **acceleration** at time t, $a(t)$, is defined by $a(t) = v'(t) = s''(t)$, provided $s''(t)$ exists.

In problems concerning rectilinear motion a positive direction must be given for the motion. This direction is generally chosen to be upward if the motion is along a vertical line. If the motion is along a horizontal line, the positive direction is usually assumed to be to the right.

Suppose a particle moves along a straight line so that its position at time $t \geq 0$ is given by

$$s(t) = t^3 - 12t^2 + 36t,$$

where t is measured in minutes (min) and $s(t)$ in centimeters (cm). Find the velocity and acceleration of the particle and describe its motion.

The velocity in cm/min at any time t is

$$v(t) = s'(t) = 3t^2 - 24t + 36,$$

and the acceleration in cm/min^2 is

$$a(t) = v'(t) = s''(t) = 6t - 24.$$

We can use $v(t)$ to determine the direction of the motion. Since

$$v(t) = 3(t^2 - 8t + 12) = 3(t-6)(t-2),$$

$v(2)$ and $v(6)$ are both zero. The particle is instantaneously stopped at $t = 2$ min and at $t = 6$ min.

When t is less than 2 or greater than 6, $v(t) > 0$, so $s(t)$ is increasing and the motion is to the right. When t is between 2 and 6, $v(t) < 0$ and the motion is to the left.

Calculating $s(t)$ for $t = 0$, $t = 2$, $t = 6$, and $t = 8$, we can sketch the pattern of motion of the particle as shown in Figure 3.2. Although the actual path of the particle lies along a line s, we show the motion above this line so that overlapping paths can be distinguished.

Since $a(t) = 6t - 24$ is negative when $0 \leq t < 4$, the velocity of the particle is decreasing during that time. The velocity is increasing when $t > 4$. At $t = 4$, the velocity is not changing since $v'(4) = a(4) = 0$.

A ball thrown upward from the earth with an initial velocity of 88 ft/sec is $s(t) = 88t - 16t^2$ ft above the earth t sec after it has been thrown. This equation is valid from the time the ball is thrown until the time it returns to earth.
Find:

(a.) the velocity and acceleration of the ball at any time t;

(b.) how many seconds it takes the ball to reach its highest point;

(c.) how high the ball goes;

(d.) how many seconds it takes the ball to reach the ground; and

(e.) the velocity of the ball when it hits the ground.